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Optimal path and directed percolation

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An efficient transfer matrix technique is introduced to study directed optimal paths in two and three dimensions. The roughness exponent ζ is 0.6325 ± 0.0007 for the two-dimensional case and $\zeta = 0.555 \pm 0.008$ for the three-dimensional one, in agreement with the recent conjecture $\zeta = \nu_\perp / \nu_\parallel$, where $\nu_\perp$ and $\nu_\parallel$ are the correlation length exponents of directed percolation. Exactly solvable examples are also analyzed.

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The problem of directed polymers (DP's) in random media [1] has recently attracted much attention because of its possible connections with other systems, such as domain walls in random ferromagnets [2], interface growth [3], and directed percolation (for a review see Ref. [4]). In particular, it is well known that the domain wall geometry in weakly disordered ferromagnets, the dynamics of growing interfaces governed by the Kardar-Parisi-Zhang (KPZ) [3] equation and DP's in a weakly disordered landscape are related to each other in two dimensions via the Burgers equation [4,5]. However, there are many physical systems where, in the considered length scales, the weak disorder approximation is not appropriate. This happens when physical features have a broad distribution and we will refer to this situation as the strong disorder case [6]. It has been shown that in this limit the self-affine domain walls become fractally rough with significant overhanging configurations, thus leading to a new universality class [7–9]. The nondirectedness of the domain wall configurations in the strong disorder limit makes it clear that the weak disorder connection with DP's is lost. Then the question arises about the behavior of DP's in a strongly disordered landscape and about the influence of the constraint of directness on the universality class [7].

The directed optimal path problem is defined in a $(d+1)$-dimensional lattice, where a random energy ϵ_b is assigned to each bond and is independently distributed all over the lattice. The energy of a directed path $\wp$ (steps occur along bonds which have a positive projection on the $(1,1\ldots,1)$ direction) is defined as

$$E(\wp) = \sum_{b \in \wp} \epsilon_b . \tag{1}$$

If the distribution is very broad, a useful approximation is to assume that the sum of variables from the distribution is simply equal to the largest one. Then the energy in Eq. (1) becomes

$$E_m(\wp) = \max_{b \in \wp} \epsilon_b . \tag{2}$$

The definition (2) allows us to deal with a broad distribution of disorder in the thermodynamic limit. In fact, choosing (1), any distribution of the disorder flows under renormalization to a Gaussian, corresponding to the weak disorder limit [2]. The ground state configuration is then the one corresponding to the path that minimizes expression (2), and it is called the *optimal path*, with energy E_c [7,8,10]. By definition, when two paths have the same energy E_c, the one which has the lower second maximum is considered to be optimal. If the paths are still degenerate the next maximum has to be analyzed and the procedure must be repeated until the degeneration is completely removed.

Recently, Roux and Zhang [11] gave an argument stating that the optimal path lives on a directed percolation cluster, and that the ground state energy corresponds to the bond percolation threshold. Moreover, due to degeneracy arguments, they claimed that the characteristic exponents should also correspond to those of directed percolation. Our aim is to show both analytically and numerically that this is the case.

We now show that on the Bethe lattice and on hierarchical lattices the directed percolation thresholds and the energy of the optimal path coincide. The key of the proof is to find the proper quantities to be analyzed. For directed percolation this quantity is the probability P of a site to be connected to the infinite percolating cluster. On the Bethe lattice bonds are oriented from a generation to the next one. Two sites are connected if an oriented path of occupied bonds joins them. Then, calling p the probability that the bond is occupied, the self-consistency equation for P is

$$P(p) = 1 - [1 - pP(p)]^{z-1} \tag{3}$$

where z is the number of nearest neighbors on the lattice. From this equation it can be found that the percolation threshold is $p_c = 1/(z-1)$. [Notice that this threshold is the same as in the case of ordinary (undirected) percolation [13].] In the case of the optimal path the key quantity is the probability of a path with an energy less than E to propagate *ad infinitum*, never traversing bonds of higher energy. Assigning a uniform random energy between 0 and 1 to each bond, the self-consistency equation we get for the path is the same as for percolation, just changing the probability p that a bond is occupied with the energy E of the path; then of course the energy of the optimal path (the smallest energy for which there is a nonzero probability of propagation) is equal to the percolation threshold. The identification of the propagation probability as the key one to study the optimal path is

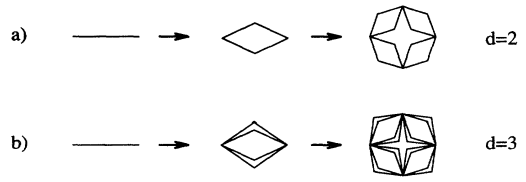

FIG. 1. Hierarchical lattices considered in this paper and corresponding to (a) $d=2$ and (b) $d=3$.

very important, because it can be used also on the diamond hierarchical lattice [12] shown in Fig. 1. In fact, we can perform an exact renormalization group on this lattice: the probability that a path of energy less than E propagates through a bond must be the same as the probability of propagating through the diamond.

In view of comparing our numerical results for d-dimensional cubic lattices, we perform a general renormalization group (RG) transformation where the single bond has to be substituted by an N-sided diamond, where $N=2^{d-1}$ (see Fig. 1). Then we can write a general expression for the RG equation for the probability $p(E)$ that a bond has an energy less than E. The renormalized probability is then

$$p'(E)=1-[1-p^2(E)]^N \tag{4}$$

with initial condition $p(E)=E$.

This is exactly the same recursion as the one for p for directed percolation (which again is the same as ordinary percolation) on these lattices. For $d=2$ and $d=3$ we find $E_c=0.6180\ldots$ and $E_c=0.2818\ldots$, respectively (notice that in the limit $d\to\infty$ $E_c\sim 1/N$, a behavior different from the Bethe lattice result which is the correct one at order $1/d^2$ with $z=2d$).

We now discuss the numerical simulations on square and cubic lattices using transfer matrix techniques. We explain the algorithm on a two-dimensional lattice, its generalization to any dimension being straightforward. In Fig. 2 a portion of the lattice is shown, where the diagonal represents the time axis. We are interested in the optimal path connecting a site at $t=t_i$ to an arbitrary site at time $t_f>t_i$. Rows t and $t+1$ of the lattice (Fig. 3) are connected by $2t$ bonds, and a uniform random value between 0 and 1 is assigned to them.

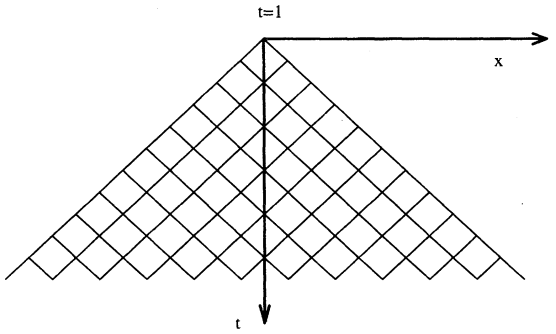

FIG. 2. Directed square lattice used for numerical simulations.

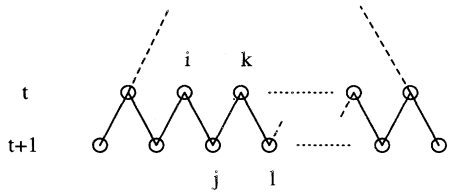

FIG. 3. Detail of the square lattice.

The energies of the optimal paths connecting each site of row $t+1$ to a site of the top line are obtained via the following algorithm:

$$E_j=\min[\max(E_i,\epsilon_{ij}),\ \max(E_k,\epsilon_{kj})] \tag{5}$$

where E_i, E_k are the energies of the optimal paths connecting the first row to sites i and k, respectively; ϵ_{ij} and ϵ_{kj} are the energies of the bonds linking sites i and j, k, and j respectively. Assigning a zero energy to the sites of the first row, and then applying the algorithm (5), the *true* optimal path energy is the minimum of the energies of the sites of the last row. In order to obtain a homogeneous set of data, we have performed our simulation by keeping fixed the ratio t_f/t_i between the final and the initial row. For computational convenience we have chosen $t_f/t_i=1000$ and t_i ranging from $10<t_i<300$ in $d=2$ and $t_f/t_i=10$ and $10<t_i<50$ for $d=3$. We have analyzed the nonlinear sequences $E_c(t_i)$ with standard extrapolation techniques [e.g. Brezinski's θ algorithm [14,15], Bulirsch and Stoer technique (BST) algorithm [16,15]] and we obtained $E_c=0.6447\pm0.0001$ in two dimensions and $E_c=0.291\pm0.002$ in three dimensions. These results are in good agreement with the best estimates existing in the literature: $p_c=0.644\,701\pm0.000\,001$ [17,18] in $d=2$ and $p_c=0.287\,30\pm0.000\,06$ [19] in $d=3$.

We also investigate the roughness properties of the optimal path, defined as the end-to end perpendicular displacement of the path. A major difficulty in this case is that the path characterized by the energy (2) is highly degenerate, so that it is not possible to doubtless identify the end-to-end displacement of the optimal path. The sources of degeneracy are the so called *neckties*, where the incoming path, with energy E, finds more than one way to go to the bottom line without increasing its energy. One way out of this problem could be to compare the second, third energy maxima, and so on, choosing the *true* optimal path as the one with the lowest maxima at any order.

A modification of the previously described algorithm of the transfer matrix, consisting of a backward scan of the lattice [20], allows one to implement a very efficient way to avoid the problem of the high degeneracy.

In the new algorithm a two-dimensional vector is assigned to each site i: the first entry E_i is the energy of the optimal path from that site to the bottom and the second one, X_i is the transverse coordinate of the extremum of the optimal path at the final time.

The procedure starts at the bottom of the lattice where at the generic site i the energy E_i is chosen to be 0 and $X_i=x_i$, the transverse coordinate of the site i itself. Then the updating of the vector assigned to a generic site k at time

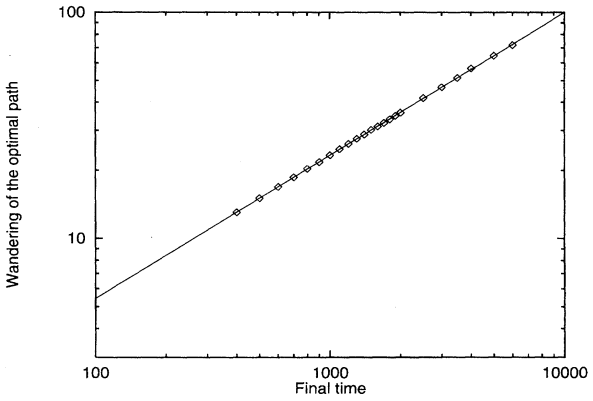

FIG. 4. Log-log plot of the wandering of the optimal path vs the final time in $d=2$.

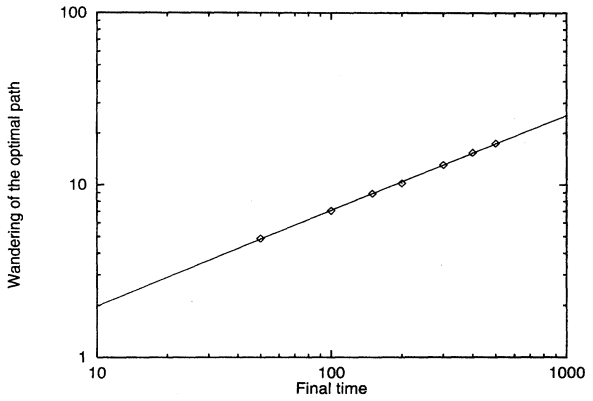

FIG. 5. Log-log plot of the wandering of the optimal path vs the final time in $d=3$.

t is done in terms of the two vectors associated with sites l and j at time $t+1$, shown in Fig. 3, according to the rules

$$E_k=\min[\max(E_j,\epsilon_{jk}),\max(E_l,\epsilon_{lk})], \tag{6}$$

$$X_k=\begin{cases} X_j & \text{if} \quad E_k=E_j \\ X_l & \text{if} \quad E_k=E_l. \end{cases} \tag{7}$$

This procedure gives the optimal path from each site at time t_i to the final time t_f. For simplicity we let the algorithm go with the time step $\Delta t=1$. The genuine Roux-Zhang conjecture states that the optimal path lives on the directed percolation cluster on an *infinite* lattice, which was the asymptotic situation for the threshold calculation setup. The setup for the calculation of the roughness is somewhat different, and the optimal path can be seen as having a fixed extreme. If the fixed extreme lies on the percolation cluster, then the Roux-Zhang argument is valid; else the path travels until it meets the directed percolation cluster: from there on it never abandons it. As a consequence, apart from an initial transient, the roughness of the path is bounded by the width of the percolating cluster. The energy, instead, is in general greater than or equal to its true value, due to an initial regime before contact with the percolation cluster is made, so that it will be different from the percolation threshold.

Using the above described algorithm, we are able to give a very precise estimate for the roughness W of the path defined as the transverse displacement of the optimal path at time t. Given the scaling law for W

$$W\sim t^{\zeta}, \tag{8}$$

the exponent ζ is related to the correlation length exponents of directed percolation via the formula [11]

$$\zeta=\frac{\nu_\perp}{\nu_\parallel}. \tag{9}$$

Indeed in directed percolation, at the critical threshold p_c, the percolating cluster is anisotropic and characterized by two correlation lengths, $\xi_\parallel\sim|p-p_c|^{-\nu_\parallel}$, in the time direction, and $\xi_\perp\sim|p-p_c|^{-\nu_\perp}$ in the x direction [21]. Identifying $\xi_\parallel$ with t and $\xi_\perp$ with W we get (8) and (9).

Figure 4 shows the log-log plot of W versus t up to $t=6000$ for the two-dimensional case. From the best fit we get $\zeta=0.6325\pm0.0007$ to be compared with the best result known [18] $\zeta=0.6326\pm0.0002$.

Figure 5 shows the corresponding data for $d=3$. In this case $\zeta=0.555\pm0.008$ is obtained (close even if not in agreement as in the $d=2$ case to the previous results $\zeta=0.567\pm0.008$ obtained in [19] using Monte Carlo simulations).

In conclusion we have given quite convincing numerical and analytical evidence that the directed optimal path [9] has a roughness exponent related to correlation length indices of directed percolation [11]. For this purpose an efficient transfer matrix algorithm was necessary [20] which allowed excellent estimates of exponents within 0.01% and 0.2% in $d=2$ and $d=3$, respectively. Since directed percolation has an upper critical dimension $d_c=4$ [22] we thus expect this is also the case for the directed extremal optimal path (i.e., when $d>d_c$, $\zeta=1/2$ like standard random walks). Surprisingly this seems also the case for directed polymers in the weak disorder approximation [23,24].

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